JOURNAL OF APPROXIMATION THEORY 9, 278–286 (1973)

Approximation by Order Intervals

J. T. MARTI

Seminar für Angewandte Mathematik, Eidgenössische Technische Hochschule, 8006 Zürich, Switzerland

Communicated by E. W. Cheney

Received November 3, 1971

1. INTRODUCTION

For many applications in approximation theory it is desirable to approximate a function f in a function space E by two other functions, one constituting an upper and the other a lower bound for f, where both bounds are supposed to be in a finite dimensional linear subspace V of E. Almost necessarily, such a theory has to be formulated in the framework of ordered normed vector lattices. To work with such structures is barely a restriction since most of the spaces used in approximation theory are normed vector lattices anyway. Let us only mention, as examples, the space $C(X)$ of continuous real functions on a compact Hausdorff space X, or the real L_p-spaces with $1 \leqslant p < \infty$. So we are led to the problem of approximating an element f by order intervals which contain f and which are of minimal length.

In this note, we derive a very general necessary and sufficient condition on the subspace V of a normed vector lattice E, in order that best interval approximations exist for every element of E. As in the case of best approximating functions it can be shown that the set of best interval approximations is convex, from which the uniqueness of best interval approximations follows for strictly convex normed vector lattices. For $E = C(X)$ we obtain a necessary condition for an order interval to be a best interval approximation. Conversely, if $X = [0, 1]$ and V is a Haar subspace of $C(X)$ we derive also an easy applicable sufficient condition for an order interval to be a best interval approximation. It is a well known fact that best one-sided polynomial approximations in $C[0, 1]$ are just translates of best Tchebycheff approximations, and in this case it turns out that the order interval between the upper and lower one-sided approximations is a best interval approximation. Using the above results it is shown that γ is a best (polynomial) interval approximation for a function f in $C[0, 1]$ if and only if both $\inf(\gamma)$ and sup(γ) constitute best (lower and upper, respectively) one-sided polynomial approximations for f.

The author acknowledges Corollary 8 and the necessity-part of the proof of Proposition 7 which both have been proposed by the referee.

2. Best Interval Approximations in Normed Vector Lattices

Let E be a normed real vector lattice (throughout we use the terminology of [7]). Without loss of generality we may assume that the positive cone K in E is closed since the ordering of E introduced by the closure of K again generates a normed vector lattice. If V is a finite dimensional subspace of E, let Γ_V be the set of all order intervals $[f, g]$ in E with $f, g \in V$, and for $\gamma \in \Gamma_V$ let $\mu(\gamma) = \| \sup(\gamma) - \inf(\gamma)\|$. If $\gamma = [f, g]$ one obviously has $\mu(\gamma) = \|f - g\|$. If f is in E we now define $M_V(f) = \{\gamma \in \Gamma_V : f \in \gamma\}$ and call $\gamma \in M_V(f)$ a *best interval approximation* for f if $\mu(\gamma) = \inf\{\mu(\gamma'): \gamma' \in M_V(f)\}$.

Theorem 1. *The sets $M_V(f)$ $(f \in E)$ are all nonempty if and only if E has an order unit in V.*

Proof. Let all $M_V(f)$ be nonempty. If $\{g_1, ..., g_n\}$ is a basis for V, take $g = \sup\{| g_1 |, ..., | g_n |\}$. Since $M_V(g)$ is nonempty, there are real numbers α_i such that $g \leqslant \sum_{i=1}^{n} \alpha_i g_i$. Take $b \in [0, 1]$ such that b is not in the spectrum of the matrix $[a_{ij}]$ given by $a_{ij} = \alpha_j$, $1 \leqslant i, j \leqslant n$. We claim that the set $\{h_1, ..., h_n\} \subset K$ given by $h_i = \sum_{j=1}^{n} \alpha_j g_j - b g_i$ forms a basis for V: If β_i are real numbers such that $\sum_{i=1}^{n} \beta_i h_i = 0$ then one has $\sum_{j=1}^{n} [\sum_{i=1}^{n} (\alpha_j - b\delta_{ij})\beta_i] g_i = 0$, hence, $\sum_{i=1}^{n} (a_{ij} - b\delta_{ij}) \beta_i = 0$, $j = 1, ..., n$. But the definition of b then implies that all β_i vanish which shows that the h_i are linearly independent. Suppose now that f is any function in E. From the initial assumption it follows that there are real numbers β_i such that $f \leqslant \sum_{i=1}^{n} \beta_i h_i$. Thus, $f \leqslant (\max_{1 \leqslant i \leqslant n} | \beta_i |) \sum_{i=1}^{n} h_i$ which proves that $\sum_{i=1}^{n} h_i$ is an order unit of E (in V).

Conversely, if E has an order unit in V, say e, then for each $f \in E$ there are positive numbers a and b such that $f \leqslant be$ and $-f \leqslant ae$. Hence, $f \in [-ae, be]$ so that $M_V(f)$ is nonempty.

Next, let $P_V(f) \subset M_V(f)$ denote the set of all best interval approximations for f.

Corollary 2. *The sets $P_V(f) (f \in E)$ are all nonempty if and only if E has an order unit in V.*

Proof. The necessity follows immediately from the theorem. Thus, let f be in E, and let $M_V(f)$ be nonempty. Define $d = \inf\{\mu(\gamma'): \gamma' \in M_V(f)\}$.

If $\{\gamma_k\} \subset M_V(f)$ is a sequence such that $d = \lim_k \mu(\gamma_k)$ it follows that $\| \sup(\pm\gamma_k)\| \leqslant \| \sup(\pm\gamma_k) \mp f\| + \|f\| \leqslant \mu(\gamma_k) + \|f\|$. Then, since $\{\mu(\gamma_k)\}$ is bounded and V is finite dimensional, there is a $\gamma \in \Gamma_V$ and a subsequence $\{\gamma_k'\}$ of $\{\gamma_k\}$ such that $\lim_k \sup(\pm\gamma_k') = \sup(\pm\gamma)$. Since K is closed and $\{\sup(\pm\gamma_k') + f\} \subset K$ one has $\gamma \in M_V(f)$. Finally, $\mu(\gamma) = \lim_k \mu(\gamma_k') = d$ shows that $\gamma \in P_V(f)$.

LEMMA 3. *The sets $P_V(f)(f \in E)$ are convex.*

Proof. Let $\gamma_1, \gamma_2 \in P_V(f)$ and $t_1, t_2 \in [0, 1]$ with $t_1 + t_2 = 1$. As a consequence of the fact that E, as a vector lattice, possesses the decomposition property we obtain

$$\begin{aligned}\sum_i t_i\gamma_i &= \sum_i t_i \inf(\gamma_i) + \sum_i [0, t_i \sup(\gamma_i) - t_i \inf(\gamma_i)]\\ &= \sum_i t_i \inf(\gamma_i) + \left[0, \sum_i (\sup(\gamma_i) - \inf(\gamma_i))\right]\\ &= \left[\sum_i t_i \inf(\gamma_i), \sum_i t_i \sup(\gamma_i)\right].\end{aligned}$$

Moreover, from $\sup(\pm\gamma_i) \mp f \in K$, $i = 1, 2$, we have

$$\sum_i t_i \sup(\pm\gamma_i) \mp f \in K,$$

and, hence, $\sum_i t_i\gamma_i \in M_V(f)$. It remains to verify that with $d = \mu(\gamma_1)$ one has

$$\begin{aligned}\mu\left(\sum_i t_i\gamma_i\right) &= \left\| \sum_i t_i(\sup(\gamma_i) - \inf(\gamma_i))\right\|\\ &\leqslant \sum_i t_i\mu(\gamma_i) = d,\end{aligned}$$

so that finally $\sum_i t_i\gamma_i \in P_V(f)$.

An immediate consequence of the lemma is the following uniqueness theorem.

THEOREM 4. *If E is strictly convex and has an order unit in V, then each of the sets $P_V(f)(f \in E)$ contains exactly one element.*

Proof. Since for $f \in V$ the statement is trivial, let $f \notin V$. By Corollary 2,

$P_V(f)$ is nonempty. Assume now that there are two distinct order intervals γ_1 and γ_2 in $P_V(f)$. Due to the lemma one has

$$\begin{aligned}
0 < \mu(\gamma_1) &= \mu(\gamma_2) = \mu(\tfrac{1}{2}\gamma_1 + \tfrac{1}{2}\gamma_2) \\
&= \tfrac{1}{2}\mu(\gamma_1)\,\mu[\mu(\gamma_1)^{-1}\,\gamma_1 + \mu(\gamma_2)^{-1}\,\gamma_2] \\
&= \mu(\gamma_1)\tfrac{1}{2}\,\|\,\mu(\gamma_1)^{-1}\,[\sup(\gamma_1) - \inf(\gamma_1)] + \mu(\gamma_2)^{-1}\,[\sup(\gamma_2) - \inf(\gamma_2)]\| \\
&< \mu(\gamma_1),
\end{aligned}$$

and this contradiction shows that $P_V(f)$ is a singleton.

We note that the above result holds whenever $E = L_p(S, \Sigma, \nu)$ $(1 < p < \infty)$ where (S, Σ, ν) is a finite positive measure space and E is ordered by the positive cone K of all ν-essentially nonnegative functions in $L_p(S, \Sigma, \nu)$. Concerning the space $L_1[0, 1]$ of all Lebesgue integrable functions on $[0, 1]$ it is clear that for $f \in L_1[0, 1]$, $\gamma \in M_V(f)$ is in $P_V(f)$ if and only if both $\inf(\gamma)$ and $-\sup(\gamma)$ are best one-sided approximations for f and $-f$, respectively. Therefore, if V is a Haar subspace (cf. Section 3) in $L_1[0, 1]$, it follows as a direct application of the results of DeVore and Bojanic [1, 3] that the best interval approximations in $M_V(f)$ for $f \in L_1[0, 1]$ are unique whenever f is in $C^1[0, 1]$ and V is spanned by a Haar system in $C^1[0, 1]$. Moreover, if $1 \leqslant p < \infty$ and if E_p is the (dense) normed vector lattice of all ν-essentially bounded functions in $L_p(S, \Sigma, \nu)$, it is clear that E_p contains order units, e.g. the characteristic function of S. As a consequence of Corollary 2 we then obtain for $E = E_p$: If V contains an order unit of E_p, then for every $f \in E_p$ there exists a best interval approximation for f in $M_V(f)$.

3. Best Interval Approximations in $C(X)$

Here, we treat the special case where E is the Banach space $C(X)$ of all continuous real functions on a compact Hausdorff space X, where $C(X)$ has the uniform norm topology and the order structure induced by $\mathbb{R}^X$. If $f \in C(X)$ and $\gamma \in M_V(f)$, we define the sets

$$\begin{aligned}
Z_0(\gamma) &= \{t \in X\colon [\sup(\gamma) - \inf(\gamma)](t) = \mu(\gamma)\}, \\
Z_1(\gamma) &= \{t \in X\colon [\sup(\gamma) - f](t) = 0\},
\end{aligned}$$

and

$$Z_2(\gamma) = \{t \in X\colon [f - \inf(\gamma)](t) = 0\}.$$

The next theorem yields necessary conditions for $\gamma \in M_V(f)$ to be a best interval approximation for f.

THEOREM 5. *Let V contain an order unit of $C(X)$ and let f be in $C(X)\backslash V$ and γ in $M_V(f)$. Then γ is not in $P_V(f)$ whenever*

(i) *$Z_1(\gamma) \cup Z_2(\gamma)$ is empty or*

(ii) *there are functions u_1, $u_2 \in V$ satisfying*

$$\begin{aligned} u_1(t) - u_2(t) &\leqslant 0, & t &\in Z_0(\gamma), \\ u_1(t) &> 0, & t &\in Z_1(\gamma), \\ u_2(t) &< 0, & t &\in Z_2(\gamma). \end{aligned}$$

Proof. Since case (i) is obvious, as well as the case where $Z_1(\gamma)$ or $Z_2(\gamma)$ are empty, we assume that both $Z_1(\gamma)$ and $Z_2(\gamma)$ are nonempty and that the inequalities of (ii) are true. Without loss of generality we may assume that $\| u_1 \| = \| u_2 \| = \| e \| = 1$, where e is an order unit of $C(X)$ in V. Let

$$d = \tfrac{1}{2} \inf\{| u_i(t)|: t \in Z_i(\gamma), i = 1, 2\}.$$

Since each u_i is continuous and the $Z_i(\gamma)$ are compact one has $d > 0$. Next, if $Z_0(\gamma) = X$, define $U_0 = X$, otherwise there exists an open neighborhood U_0 of $Z_0(\gamma)$ in X such that

$$u_1(t) - u_2(t) \leqslant d \inf\{e(t): t \in X\}/2, \qquad t \in U_0 .$$

Since $f \notin V$ we obviously have $Z_1(\gamma), Z_2(\gamma) \neq X$. Consequently, there are open neighborhoods U_i of $Z_i(\gamma)$ in X such that

$$(-1)^{i-1} u_i(t) > d, \qquad t \in U_1, \qquad i = 1, 2.$$

Let now a, b, and δ be the positive constants given by

$$\begin{aligned} a &= \inf\{[\sup(-(-1)^i\gamma) + (-1)^i f](t): t \in X \setminus U_i , i = 1, 2\}, \\ b &= \begin{cases} \mu(\gamma) - \sup\{[\sup(\gamma) - \inf(\gamma)](t): t \in X \setminus U_0\}, & U_0 \neq X, \\ a, \quad U_0 = X, \end{cases} \\ \delta &= \min\{a, b/2\}/(2 + d), \end{aligned}$$

and define g_1 and g_2 in V by

$$\begin{aligned} g_1 &= \sup(\gamma) + \delta u_1 - \delta de, \\ g_2 &= \inf(\gamma) + \delta u_2 . \end{aligned}$$

From

$$\begin{aligned} g_1 - f &= \sup(\gamma) - f + \delta(u_1 - de), \\ f - g_2 &= f - \inf(\gamma) - \delta u_2 , \end{aligned}$$

we then get

$$(g_1 - f)(t) \geqslant \delta d(1 - e(t)) \geqslant 0, \qquad t \in U_1,$$
$$(g_1 - f)(t) \geqslant a - \delta(1 + d) \geqslant 0, \qquad t \in X \setminus U_1,$$
$$(f - g_2)(t) \geqslant \delta d \geqslant 0, \qquad t \in U_2,$$

and

$$(f - g_2)(t) \geqslant a - \delta \geqslant 0, \qquad t \in X \setminus U_2.$$

Therefore, $f \in [g_2, g_1]$ and so $[g_2, g_1] \in M_V(f)$. We finally get

$$0 \leqslant (g_1 - g_2)(t) = [\sup(\gamma) - \inf(\gamma)](t) + \delta(u_1 - u_2 - de)(t)$$
$$\leqslant \mu(\gamma) - \delta d \inf\{e(t): t \in X\}/2, \qquad t \in U_0$$

and similarly

$$0 \leqslant (g_1 - g_2)(t) \leqslant \mu(\gamma) - b + \delta(2 + d) \leqslant \mu(\gamma) - b/2, \qquad t \in X - U_0.$$

Hence, $\mu([g_2, g_1]) < \mu(\gamma)$ and this completes the proof.

The following definitions we need to derive (for the special case $X = [0, 1]$) a sufficient condition for $\gamma \in M_V(f)$ to be in $P_V(f)$, a condition that may easily be used to check whether an order interval containing f is a best interval approximation for f.

V is said to be a *Haar subspace* of $C(X)$ if for every set of n distinct points $t_1, ..., t_n$ of X and every real n-tuple $(y_1, ..., y_n)$ the interpolation problem $y_i = g(t_i)$, $i = 1, ..., n$ has a unique solution g in V. Furthermore, let B be the Boolean algebra of all ordered triples whose elements are the numbers 0 or 1 (B with the usual lattice structure), and let $\mathscr{B}$ be the set of all subsets of B. If $f \in C[0, 1]$ and if γ is a fixed order interval in $M_V(f)$, we define the function $W_\gamma: [0, 1] \to \mathscr{B}$ by

$$W_\gamma(t) = \{(x_0, x_1, x_2) \in B \setminus \{(1, 1, 1)\}: x_i = 1 \text{ whenever } 0 \leqslant i \leqslant 2 \text{ and } t \in Z_i(\gamma)\}.$$

Next, let $H(\gamma)$ be the set of all functions F: $[0, 1] \to B$ such that $F(t) = (F_0(t), F_1(t), F_2(t)) \in W_\gamma(t)$, $t \in [0, 1]$. If $\mathrm{Var}(g)$ denotes the total variation of a real function g on $[0, 1]$, we finally define

$$\mathrm{Var}(F) = \max_{0 \leqslant i \leqslant 2} \mathrm{Var}(F_i), \qquad F \in H(\gamma).$$

THEOREM 6. *Let V be a Haar subspace of dimension n in $C[0, 1]$. If $f \in C[0, 1] \setminus V$, then $\gamma \in M_V(f)$ is in $P_V(f)$ whenever* $\inf\{\mathrm{Var}(F): F \in H(\gamma)\} \geqslant n$.

Proof. Let γ be in $M_V(f)$ and let $\inf\{\mathrm{Var}(F): f \in H(\gamma)\} \geqslant n$. Assume that γ is not in $P_V(f)$. Then there is a $\gamma' \in M_V(f)$ such that $\mu(\gamma') < \mu(\gamma)$. We define g and h in V by

$$g = \sup(\gamma) - \sup(\gamma')$$

and

$$h = \inf(\gamma) - \inf(\gamma').$$

From this follows that for $t \in Z_0(\gamma)$, $Z_1(\gamma)$ or $Z_2(\gamma)$ one has

$$g(t) - h(t) > 0, \tag{1}$$

$$g(t) \leqslant 0, \tag{2}$$

or

$$h(t) \geqslant 0, \tag{3}$$

respectively. Let now the function G: $[0, 1] \to B$, depending on γ, g, and h, be given by $G(t) = (G_0(t), G_1(t), G_2(t))$, $t \in [0, 1]$, where $G_j(t) = 1$ if the above statement $(j + 1)$ is true and $G_j(t) = 0$ otherwise. Obviously, $G \in H(\gamma)$ so that, by hypothesis, $\mathrm{Var}(G) \geqslant n$. This implies that either $g - h$ or g or h must have at least n zeros in $[0, 1]$ (counting double zeros in $(0, 1)$ as two zeros), hence that either (i) $g = h$ or (ii) $g = 0$ or (iii) $h = 0$.

If (i) holds we have $\sup(\gamma) - \sup(\gamma') = \inf(\gamma) - \inf(\gamma')$ and from this follows that $\mu(\gamma') = \mu(\gamma)$ which is a contradiction.

If (ii) is true we have $\sup(\gamma') = \sup(\gamma)$. Assume now that $Z_0(\gamma) \cap Z_2(\gamma)$ is nonempty. Then there is a $t \in Z_0(\gamma) \cap Z_2(\gamma)$ and for this t we have $[\sup(\gamma) - \inf(\gamma)](t) = \mu(\gamma)$ and $h(t) \geqslant 0$. Hence, $\mu(\gamma') \geqslant [\sup(\gamma') - \inf(\gamma')](t) = [\sup(\gamma) - \inf(\gamma) - \inf(\gamma') + \inf(\gamma)](t) = \mu(\gamma) + h(t) \geqslant \mu(\gamma)$ which is again a contradiction to our initial assumption. One, therefore, has to investigate the remaining case, $Z_0(\gamma) \cap Z_2(\gamma) =$ empty. This last statement shows that the set $H'(\gamma)$ of all function F: $[0, 1] \to \{(1, 1, 0), (0, 1, 1)\}$ such that

$$F(t) = (1, 1, 0), \qquad t \in Z_0(\gamma)$$

and

$$F(t) = (0, 1, 1), \qquad t \in Z_2(\gamma)$$

is contained in $H(\gamma)$. Since then

$$\inf\{\mathrm{Var}(F): F \in H'(\gamma)\} \geqslant \inf\{\mathrm{Var}(F): F \in H(\gamma)\} \geqslant n$$

it follows that $g - h$ has at least n zeros in $[0, 1]$ (again counting double zeros in $(0, 1)$ as two zeros), hence, that $g = h$. The desired contradiction is reached as in (i).

The case (iii) may be treated in a completely analogous way as case (ii). We, therefore, have to conclude that $\gamma \in P_V(f)$.

EXAMPLE. Let $V \subset C[0, 1]$ be spanned by the basis $\{x_1, x_2\}$, where $x_1(t) = 1 - t$ and $x_2(t) = t^2$, $t \in [0, 1]$. It is easy to verify that $\det[x_i(t_j)] \neq 0$

if $0 \leqslant t_1 < t_2 \leqslant 1$, hence, that V is a Haar subspace of $C[0, 1]$. Let $f \in C[0, 1] \backslash V$ be the constant function $f(t) = 1$, $t \in [0, 1]$, take $g(t) = x_1(t) + x_2(t) = 1 - t + t^2$, $t \in [0, 1]$, and let $h = (4/3)g$. Since

$$\inf\{h(t): t \in [0, 1]\} = h(\tfrac{1}{2}) = \sup\{g(t): t \in [0, 1]\} = 1$$

we have $[g, h] \in M_V(f)$.

By combinatorial considerations we can see that

$$W(0) = W(1) = \{(1, 0, 1)\},$$
$$W(\tfrac{1}{2}) = \{(1, 1, 0), (0, 1, 0), (0, 1, 1)\},$$

and

$$W(t) = B \backslash \{(1, 1, 1)\} \qquad \text{otherwise.}$$

Therefore, $\inf\{\mathrm{Var}(F): F \in H([g, h])\} = 2$. Hence, $[g, h]$ is a best interval approximation for f in $M_V(f)$.

4. Best Interval and One-Sided Polynomial Approximations

If $f \in C[0, 1]$ we call $g \in V$ a best *one-sided approximation* for f if $g \leqslant f$ and $\| f - g \| = \inf(\| f - g' \|: g' \in f - K\}$. If V is the subspace of all algebraic polynomials of degree $n - 1$ in $C[0, 1]$ it is known [2] that $g \in V$ is a best one-sided approximation for f in $C[0, 1]$ if and only if $g + \inf\{\| f - g' \|: g' \in V\}e$ is a best Tchebycheff approximation for f, where e is the characteristic function of $[0, 1]$. We can show that in this special case of polynomial approximations, the best interval approximations may be determined from a known Tchebycheff approximation by mere translation.

Proposition 7. *Let V be the subspace of all algebraic polynomials of degree $n - 1$ in $C[0, 1]$ and f a function in $C[0, 1]$. Then $\gamma \in P_V(f)$ if and only if $\gamma = [g, g + \| f - g \| e]$, where $g \in V$ is a best one-sided approximation for f.*

Proof (Sufficiency). Let g be a best one-sided approximation for $f \in C[0, 1] \backslash V$. Then, obviously, $\gamma = [g, g + \| f - g \| e] \in M_V(f)$ with $\mu(\gamma) = \| f - g \|$. By the remark preceding the proposition, $\mu(\gamma) = 2 \inf\{\| f - g' \|: g' \in V\}$ and $g + \frac{1}{2}\mu(\gamma)e$ is a best Tchebycheff approximation for f. Thus, $g + \frac{1}{2}\mu(\gamma)e - f$ alternates n times on $[0, 1]$ from which we infer the existence of an integer k and of points $0 \leqslant t_1 < \cdots < t_{n+1} \leqslant 1$ such that

$$t_i \in Z_{j(i)}(\gamma), \qquad 1 \leqslant i \leqslant n + 1,$$

where $j(i) = \frac{1}{2}(3 - (-1)^{i+k})$. Obviously, $z(\gamma) = [0, 1]$ so that each $F \in H(\gamma)$ satisfies

$$F(t_i) = \begin{cases} (1, 1, 0), & i + k \text{ even}, \\ (1, 0, 1), & i + k \text{ odd}. \end{cases}$$

Accordingly, $\mathrm{Var}(f) \geqslant n$ for all $F \in H(\gamma)$. Together with Theorem 6 this shows that $\gamma \in P_V(f)$.

(Necessity). Let γ be in $P_V(f)$, where $f \in C[0, 1]$. Again by the above remark there is a best one-sided approximation $g \in V$ for f (which is unique since the best Tchebycheff approximations are unique). From the sufficiency part of the proof we know that $[g, g + \|f - g\| e] \in P_V(f)$. Hence, $\|f - g\| = \mu(\gamma)$ and so

$$0 \leqslant f - \inf(\gamma) \leqslant \mu(\gamma)e = \|f - g\| e.$$

Thus,

$$\|f - \inf(\gamma)\| \leqslant \|f - g\|,$$

which implies that $\inf(\gamma)$ is a best one-sided approximation for f in V. Since such an approximation is unique we have $\inf(\gamma) = g$. In a similar way, $\sup(\gamma) = g + \|f - g\| e$ which finally proves the proposition.

From the remark preceding Proposition 7 and the uniqueness of best Tchebycheff approximations we immediately have the following corollary.

COROLLARY 8. *Under the hypothesis of Proposition 7 the set $P_V(f)$ contains exactly one element.*

REFERENCES

1. R. BOJANIC AND R. DEVORE, On polynomials of best one-sided approximation, *Enseignement Math.* **12** (1966), 139–164.
2. L. COLLATZ, Einseitige Tschebyscheff-Approximation bei Randwertaufgaben, "Proceedings of the International Conference on Constructive Function Theory," Warna, 1970.
3. R. DEVORE, One-sided approximation of functions, *J. Approximation Theory* **1** (1968), 11–25.
4. G. FREUD, Über einseitige Approximation durch Polynome, I. *Acta Math. Acad. Sceged* **16** (1955), 12–28.
5. G. FREUD, Some results on one-sided approximation, *Math. Scand.* **5** (1957), 276–284.
6. T. GANELIUS, On one-sided approximation by trigonometrical polynomials, *Math. Scand.* **4** (1956), 247–258.
7. A. L. PERESSINI, "Ordered Topological Vector Spaces," Harper and Row, New York, 1967.